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Abstract. In this paper we begin with the construction of a generalized Segal-Bargmann transform related to every root system with finite reflection group G. To do so, we introduce a Hilbert space $\mathscr{F}_k(\mathbb{C}^N)$ of holomorphic functions with reproducing kernel equal to the Dunkl kernel. Moreover, by means of an $\mathfrak{sl}(2)$-triple, we prove the branching decomposition of $\mathscr{F}_k(\mathbb{C}^N)$ as a unitary $G \times \widetilde{SL(2,\mathbb{R})}$-module. Further applications of the $\mathfrak{sl}(2)$-triple to the Dunkl theory are given. This paper is a survey of recent results in [B-Ø2] and [B-Ø3], and it also contains new results.

1. Introduction

Around 1928, in [Fo], Fock has introduced a Hilbert space of holomorphic functions on $\mathbb{C}^N$ which are square integrable with respect to the Gaussian measure $\exp(-\|z\|^2)dz$, where $\|z\|^2 = \sum_{i=1}^N z_i\overline{z_i}$. These spaces are nowadays known as the Fock spaces. After Bargmann's elegant paper [Ba], the Fock spaces have attracted much interest and have played an important role in a number of developments, specially in physics and mathematical physics. The remarkable invention of Bargmann is the construction of a unitary map from the Schrödinger model to the Fock model intertwining the action of the Heisenberg group. This idea also appeared in the work of Segal [Seg], done independently at about the same time. This intertwining operator is the so-called Segal-Bargmann transform. Since then, the study of several generalizations of the classical Segal-Bargmann transform has been pursued in many different settings. See for instance [Gr-M], [Ó-Ø], [Ha], and [Da-Ó-Z].

While the theory of Segal-Bargmann transform has a long and rich history, the growing interest in the theory of special functions associated with Coxeter groups is comparably recent. After the important contribution by Heckman and Opdam in the area of special functions related to root systems, the subject has attracted much interest and there has been a rapid development in this subject during the last few years. Around the 90s, C. Dunkl introduced a family of differential-difference operators associated with Coxeter groups on finite dimensional Euclidean spaces. These operators are nowadays know as Dunkl operators. They are parameterized deformations of the ordinary derivatives, for which it is still possible to study the spectral problem and develop the theory of the corresponding Fourier transform, called the Dunkl transform. Among the broad literature in this area, we refer to [D1], [J], [H], [O], [R2], and [T].

In this survey we will first present results from [B-Ø2] (see section 2 below), where we investigate a generalization of both, the Fock spaces and the Segal-Bargmann transform in the setting of Coxeter groups and Dunkl operators. The motivation for studying the Segal-Bargmann transform is to exhibit some relationships between Dunkl's theory and its applications in the Schrödinger model and in the Fock model which includes the study of the Calogero-Moser systems, and the Dunkl transform. We will also prove that one can develop an analogue theory of Howe dual pairs to obtain the branching decomposition of the generalized Fock space. It turns out that there exists an $\mathfrak{sl}(2)$-triple which gives rise to a unitary representation of the universal covering $\widetilde{SL(2,\mathbb{R})}$ that is an analogue to the

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classical metaplectic representation of $Mp(2,\mathbb{R})$. By means of this representation, we prove in section 3 a Bochner formula for the Dunkl transform, and we investigate the validity of Huygens' principle for wave equations for the Dunkl-Laplacian operators. To do so, we adapt R. Howe's method for the Euclidean Fourier transform, and for the classical wave equation, respectively (cf. [Ho1]). We close section 3 by proving a Harish-Chandra type restriction theorem for the Dunkl transform. That is, there exists a Harish-Chandra type integral which intertwines the Fourier transform on Cartan motion groups and the Dunkl transform. Our argument uses M. Vergne's approach for proving the Rossmann-Kirillov character formula [Ve]. We refer to [B-Ø3] for complete results on Huygens' principle.

The main results of the present paper are: Theorems 2.5, 2.7, 2.11, 3.1, 3.4, and 3.6.

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2. Fock spaces and Segal-Bargmann transforms

Let $\langle\cdot,\cdot\rangle$ be the standard Euclidean scalar product in $\mathbb{R}^N$, as well as its bilinear extension to $\mathbb{C}^N\times\mathbb{C}^N$. For $\alpha\in\mathbb{R}^N\setminus\{0\}$, let r_α be the reflection on the hyperplane $\langle\alpha\rangle^\perp$ orthogonal to α

$$r_\alpha(x) := x - 2\frac{\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle}\alpha, \qquad x\in\mathbb{R}^N.$$

Let $\mathscr{R}$ be a reduced root system, i.e. $\mathscr{R}\cap\mathbb{R}\alpha=\{\pm\alpha\}$ for all $\alpha\in\mathscr{R}$ and $r_\alpha(\mathscr{R})=\mathscr{R}$. Henceforth, we will normalize $\mathscr{R}$ in the sense that $\langle\alpha,\alpha\rangle=2$. This simplifies formulas, with no loss of generality for our purposes.

A Coxeter group G is a finite subgroup of the orthogonal group $O(N)$ generated by the reflections $\{r_\alpha\,|\,\alpha\in\mathscr{R}\}$. Note that Coxeter groups generalize Weyl groups since there is no additional crystallographic condition for $\mathscr{R}$.

A multiplicity function on $\mathscr{R}$ is a G-invariant function $k:\mathscr{R}\to\mathbb{C}$. We set $\mathscr{K}^+$ to be the set of multiplicity functions $k=(k_\alpha)_{\alpha\in\mathscr{R}}$ such that $k_\alpha\geq 0$ for all α, and we let $\mathscr{R}^+$ be a choice of positive roots in $\mathscr{R}$.

Around 1990, C. Dunkl defined a family of first order differential-difference operators that play the role of the usual partial differentiation. Dunkl's operators are defined by

$$T_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha\langle\alpha,\xi\rangle\frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle}, \qquad f\in\mathscr{C}^1(\mathbb{R}^N),$$

where ∂_ξ denotes the directional derivative corresponding to ξ. We refer to [D1] for more details on Dunkl operators. In particular, for any orthonormal basis $\{\xi_i\}_{i=1}^N$ of $\mathbb{R}^N$, the Dunkl-Laplacian operator $\Delta_k := \sum_{i=1}^N T_{\xi_i}^2(k)$ can be written as

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in\mathscr{R}^+} k_\alpha\left\{\frac{\langle\nabla f(x),\alpha\rangle}{\langle\alpha,x\rangle} - \frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle^2}\right\},$$

where Δ and ∇ denote the usual Laplacian and gradient, respectively. For all i-th basis vectors ξ_i, we will use the abbreviation $T_{\xi_i}(k)=T_i(k)$.

For $k\in\mathscr{K}^+$, there exists a generalization of the usual exponential kernel $e^{\langle\cdot,\cdot\rangle}$ by means of the Dunkl system of differential equations.

Theorem 2.1. (cf. [D2], [O]) *For $k\in\mathscr{K}^+$, there exists a unique function E_k on $\mathbb{C}^N\times\mathbb{C}^N$ characterized by:*

(i) $T_\xi(k)E_k(z,w)=\langle\xi,w\rangle E_k(z,w)$; *and*

(ii) $E_k(z, 0) = 1$.

Moreover, this function satisfies

(iii) E_k *is holomorphic on* $\mathbb{C}^N \times \mathbb{C}^N$; *and*

(iv) $E_k(g_0 \cdot z, g_0 \cdot w) = E_k(z, w)$ *for all* $g_0 \in G$.

For complex-valued k, there is a detailed investigation of (i) by Opdam [O]. Theorem 2.1 is a weak version of Opdam's result. For integral-valued multiplicity function k, another proof for Theorem 2.1 can be found in [B-Ø1], by means of shift operators. (In the latter reference, we denote $E_k(z, w)$ by $G^\circ(z, k, w)$.) The function E_k is the so-called Dunkl kernel. When $k \equiv 0$, we have $E_0(z, w) = e^{\langle z, w\rangle}$ for $z, w \in \mathbb{C}^N$.

For $k \in \mathscr{K}^+$, let ω_k be the weight function on $\mathbb{R}^N$ defined by

$$\omega_k(x) := \prod_{\alpha \in \mathscr{R}^+} |\langle \alpha, x\rangle|^{2k_\alpha}.$$

Further, let

$$c_k := \int_{\mathbb{R}^N} e^{-\langle x, x\rangle/2} \omega_k(x)\, dx,$$

which is called the Macdonald-Metha-Selberg integral. The following proposition is crucial in Dunkl's theory and its applications.

Proposition 2.2. (cf. [D2]) *Let* $z, w \in \mathbb{C}^N$. *For non-negative multiplicity function* k,

$$\int_{\mathbb{R}^N} E_k(x, z) E_k(x, w) e^{-\langle x, x\rangle/2} \omega_k(x)\, dx = c_k e^{(\langle z, z\rangle + \langle w, w\rangle)/2} E_k(z, w). \tag{2.1}$$

For $z, w \in \mathbb{C}^N$, define

$$\mathbb{K}_{k,w}(z) = \mathbb{K}_k(z, w) := E_k(z, \overline{w}).$$

As k will be fixed, we will write $\mathbb{K}$ for $\mathbb{K}_k$. By Theorem 2.1, one may check that $\mathbb{K}$ is continuous and $\mathbb{K}_w$ is holomorphic for all $w \in \mathbb{C}^N$. Further, $\mathbb{K}(z, w) = \overline{\mathbb{K}(w, z)}$. Another crucial property is that $\mathbb{K}(z, w)$ is a positive definite kernel, i.e. for all $z^{(1)}, \ldots, z^{(\ell)} \in \mathbb{C}^N$ and $a_1, \ldots a_\ell \in \mathbb{C}$

$$\sum_{i,j=1}^{\ell} a_i \overline{a_j} \mathbb{K}(z^{(i)}, z^{(j)}) \geq 0.$$

These properties of $\mathbb{K}$ lead to the following result.

Theorem 2.3. (cf. [B-Ø2]) (i) *There exists a Hilbert space* $\mathscr{F}_k(\mathbb{C}^N)$ *of holomorphic functions on* $\mathbb{C}^N$, *such that* $\mathbb{K}$ *is its reproducing kernel.*

(ii) *The Hilbert space* $\mathscr{F}_k(\mathbb{C}^N)$ *contains the* $\mathbb{C}$*-algebra* $\mathscr{P}(\mathbb{C}^N)$ *of polynomial functions on* $\mathbb{C}^N$ *as a dense subspace.*

In particular, if we denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_k$ the inner product in $\mathscr{F}_k(\mathbb{C}^N)$, then

$$\langle\!\langle p, q\rangle\!\rangle_k = p(T(k))\overline{q(\bar{z})}_{\big|z=0}, \qquad \forall\, p, q \in \mathscr{P}(\mathbb{C}^N),$$

where $p(T(k))$ is the operator formed by replacing z_i by $T_i(z)$ for $1 \leq i \leq N$.

If $k \equiv 0$, $\mathscr{F}_0(\mathbb{C}^N)$ coincides with the classical Fock space. We shall call $\mathscr{F}_k(\mathbb{C}^N)$ the Fock space associated with the Coxeter G.

***Example* 2.4.** Let $\{e_1, \ldots, e_N\}$ be the standard basis of $\mathbb{R}^N$, and consider the reflection group G generated by the reflections $r_1, \ldots, r_N$ along $e_1, \ldots, e_N$, i.e. $r_j(\cdots, x_{j-1}, x_j, x_{j+1}, \cdots) = (\cdots, x_{j-1}, -x_j, x_{j+1}, \cdots)$ for $x \in \mathbb{R}^N$.

For a multi-parameter $k = (k_1, \ldots, k_N)$ such that $k_i \geq 0$, the Dunkl operators take the form

$$T_j(k) f(x) = \partial_j f(x) + k_j \frac{f(x) - f(r_j x)}{x_j}, \qquad 1 \leq j \leq N,\ x \in \mathbb{R}^N$$

The Dunkl kernel E_k is given by

$$E_k(z,w)=\prod_{j=1}^N \Gamma\Big(k_j+\frac{1}{2}\Big)\Big(\frac{z_jw_j}{2}\Big)^{1/2-k_j}\big\{I_{k_j-1/2}(z_jw_j)+I_{k_j+1/2}(z_jw_j)\big\},$$

where I_ν is the modified Bessel function of the first kind. In this example, the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ related to $G\cong(\mathbb{Z}/2\mathbb{Z})^N$ is the Hilbert space of holomorphic functions on $\mathbb{C}^N$ which are square integrable with respect to the measure

$$d\mu_k(z)=\prod_{j=1}^N \frac{|z_j|^{2k_j+1}}{\pi 2^{k_j-1/2}\Gamma(k_j+1/2)}\Big\{\mathscr{K}_{k_j-1/2}(|z_j|^2)\big|_{\text{even part}}+\mathscr{K}_{k_j+1/2}(|z_j|^2)\big|_{\text{ odd part}}\Big\}dz_j,$$

splitting functions into even and odd parts in each variable z_j. Here $\mathscr{K}_\nu$ is the Bessel function of the third kind.

The study of several generalizations of the classical Segal-Bargmann transform has a long and rich history in many different settings (cf. [Ba], [Seg], [Ó-Ø], [Ha], [Da-Ó-Z], [Z], [So]). There are many ways of computing the integral kernel appearing in the Segal-Bargmann transform and showing the unitarity of this transform. One unifying tool is the restriction principle, i.e. polarization of a suitable restriction map [Ó-Ø]. We will use this idea to construct the Segal-Bargmann transform associated with G. The main tool is the heat-kernel analysis for Coxeter groups [R1].

For $t>0$ and $z,w\in\mathbb{C}^N$, set

$$\Gamma_k(t,z,w)=\frac{1}{(2t)^{\gamma_k+N/2}c_k}e^{-(\langle z,z\rangle+\langle w,w\rangle)/4t}\,E_k\Big(\frac{z}{\sqrt{2t}},\frac{w}{\sqrt{2t}}\Big).$$

The kernel $\Gamma_k(t,z,w)$ was introduced in [R1] as a generalized heat kernel.

Let $\mathscr{L}^2(\mathbb{R}^N,\omega_k)$ be the space of $\mathscr{L}^2$-functions on $\mathbb{R}^N$ with respect to the weight function ω_k.

Let $\mathcal{R}_k$ be the restriction map $\mathcal{R}_k:\mathscr{F}_k(\mathbb{C}^N)\to\mathscr{L}^2(\mathbb{R}^N,\omega_k)$, given by

$$\mathcal{R}_kf(x):=e^{-\langle x,x\rangle/2}f(x),\quad x\in\mathbb{R}^N.$$

The map $\mathcal{R}_k$ is a closed, densely defined operator from $\mathscr{F}_k(\mathbb{C}^N)$ into $\mathscr{L}^2(\mathbb{R}^N,\omega_k)$ with dense image (see for instance [R1, Corollary 3.5]). Consider the adjoint $\mathcal{R}_k^*:\mathscr{L}^2(\mathbb{R}^N,\omega_k)\to\mathscr{F}_k(\mathbb{C}^N)$ as a densely defined operator. Since $\mathbb{K}$ is the reproducing kernel of $\mathscr{F}_k(\mathbb{C}^N)$, one can prove that for $f\in\mathscr{L}^2(\mathbb{R}^N,\omega_k)$, the integral

$$\mathcal{R}_k\mathcal{R}_k^*f(y)=c_k\int_{\mathbb{R}^N}f(x)\Gamma_k\Big(\frac{1}{2},x,y\Big)\omega_k(x)\,dx$$

converges absolutely for a.e. $y\in\mathbb{R}^N$. The function $\mathcal{R}_k\mathcal{R}_k^*f$ thus defined is in $\mathscr{L}^2(\mathbb{R}^N,\omega_k)$ and $\|\mathcal{R}_k\mathcal{R}_k^*\|\le c_k$. We can therefore define $\sqrt{\mathcal{R}_k\mathcal{R}_k^*}$ and there exists an isometry $\mathscr{B}_k$ so that $\mathcal{R}_k^*=\mathscr{B}_k\sqrt{\mathcal{R}_k\mathcal{R}_k^*}$. Since $\mathcal{R}_k=\sqrt{\mathcal{R}_k\mathcal{R}_k^*}\mathscr{B}_k^*$ and Image($\mathcal{R}_k$) is dense, it follows that $\mathscr{B}_k$ is a unitary isomorphism. We shall call $\mathscr{B}_k$ the Segal-Bargmann transform associated with G. Using the positivity of the heat kernel $\Gamma(t,x,y)$ as an operator [R1], we obtain the following integral representation of the Segal-Bargmann transform $\mathscr{B}_k$.

Theorem 2.5. (cf. [B-Ø2]) *The unitary isomorphism $\mathscr{B}_k:\mathscr{L}^2(\mathbb{R}^N,\omega_k)\to\mathscr{F}_k(\mathbb{C}^N)$ is given by*

$$\mathscr{B}_kf(z)=2^{\gamma_k+N/2}c_k^{-1/2}e^{-\langle z,z\rangle/2}\int_{\mathbb{R}^N}f(x)E_k(\sqrt{2}x,\sqrt{2}z)e^{-\langle x,x\rangle}\omega_k(x)\,dx,$$

where $\gamma_k:=\sum_{\alpha\in\mathscr{R}^+}k_\alpha$.

***Remark* 2.6.** (i) For the special case $k \equiv 0$,

$$\mathscr{B}_0 f(z) = (2/\pi)^{N/4} \int_{\mathbb{R}^N} e^{-\langle x,x\rangle + 2\langle x,z\rangle - \langle z,z\rangle/2} f(x)\, dx.$$

This compares well with the classical Segal-Bargmann transform (cf. [Fol, p. 40]).

(ii) As a differential operator

$$\mathscr{B}_k^{-1} = 2^{\gamma_k + N/2} c_k^{-1/2} e^{\langle\,\cdot\,,\,\cdot\,\rangle}\, d_2 \circ e^{-\Delta_k/2}, \tag{2.2}$$

where d_2 is the dilation operator on functions by 2.

(iii) When $N = 1$ and $G = \mathbb{Z}/2\mathbb{Z}$, Cholewinski [Ch] has investigated the Segal-Bargmann transform only in the Hilbert space of even functions in $\mathscr{F}_k(\mathbb{C})$, by employing another approach. Recently, in [Si-So] for $N = 1$ and $G = \mathbb{Z}/2\mathbb{Z}$, the authors use Cholewinski's method to obtain the Segal-Bargmann transform for $\mathscr{F}_k(\mathbb{C})$. See also [So] where the integral representation of $\mathscr{B}_k$ is obtained in the general case, using Cholewinski's approach.

The Dunkl transform, which shares many properties with the Euclidean Fourier transform, was introduced in [D2] and further studied in [J]. For our convenience, we will write the Dunkl transform as

$$\mathscr{D}_k f(\xi) = c_k^{-1} 2^{-\gamma_k - N/2} \int_{\mathbb{R}^N} f(x/2) E_k(-i\xi, x)\omega_k(x)\, dx, \quad \xi \in \mathbb{R}^N.$$

Theorem 2.7. (cf. [B-Ø2]) *The following diagram commutes*

$$\begin{array}{ccc} \mathscr{L}^2(\mathbb{R}^N, \omega_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \\ {\scriptstyle \mathscr{D}_k}\big\downarrow & & \big\downarrow{\scriptstyle (-i)^*} \\ \mathscr{L}^2(\mathbb{R}^N, \omega_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \end{array}$$

where $(-i)^* f(z) := f(-iz)$ *for* $f \in \mathscr{F}_k(\mathbb{C}^N)$.

The above theorem gives a simple alternative proof for the unitarity of the transform $\mathscr{D}_k$, which was proved earlier by Dunkl [D2] using a different approach. See also [J]. Our proof uses only the integral formula (2.1).

For $\xi \in \mathbb{C}^N$, denote by M_ξ the operator $M_\xi(f)(z) := \langle z, \xi\rangle f(z)$. Define the lowering and the raising operators on $\mathscr{L}^2(\mathbb{R}^N, \omega_k)$ by

$$A_\xi^- := \frac{1}{\sqrt{2}}(M_{2\xi} + T_\xi(k)), \qquad A_\xi^+ := \frac{1}{\sqrt{2}}(M_{2\xi} - T_\xi(k)).$$

These two operators were introduced by Rösler [R2] in connection with Rodrigues-type formulas for the eigenfunctions of the Calogero-Moser systems. Next we will see that these two operators, in the Fock model, are also the lowering and the raising operators on $\mathscr{F}_k(\mathbb{C}^N)$ in a more natural way.

Below, we will exhibit some relationships between operators on $\mathscr{L}^2(\mathbb{R}^N, \omega_k)$ and on $\mathscr{F}_k(\mathbb{C}^N)$. For an operator $\mathscr{O}$ on $\mathscr{L}^2(\mathbb{R}^N, \omega_k)$, we define the operator $\widetilde{\mathscr{O}}$ on $\mathscr{F}_k(\mathbb{C}^N)$ by

$$\widetilde{\mathscr{O}} = \mathscr{B}_k \circ \mathscr{O} \circ \mathscr{B}_k^{-1}.$$

Further, as usual, $[A, B] = AB - BA$ for $A, B \in \mathrm{End}(\mathscr{P}(\mathbb{C}^N))$.

Theorem 2.8. (cf. [B-Ø2]) *The following properties hold:*

(i) $\widetilde{T_\xi}(k) = T_\xi(k) - M_\xi$ *for* $\xi \in \mathbb{C}^N$;

(ii) $[\widetilde{T_\xi}(k), \widetilde{T_\eta}(k)] = 0$ *for* $\xi, \eta \in \mathbb{C}^N$;

(iii) $\widetilde{M_{2\xi}} = T_\xi(k) + M_\xi$ *for* $\xi \in \mathbb{C}^N$;

(vi) $[\widetilde{M_{2\xi}}, \widetilde{M_{2\eta}}] = 0$ *for* $\xi, \eta \in \mathbb{C}^N$;

(v) $[\widetilde{T_\xi}(k), \widetilde{M_{2\eta}}] = 2\langle \xi, \eta\rangle + 2\sum_{\alpha\in\mathscr{R}^+} k_\alpha \langle \alpha, \xi\rangle\langle \alpha, \eta\rangle r_\alpha$; *and*

(vi) $\widetilde{A}_\xi^- = \sqrt{2}T_\xi(k)$, *and* $\widetilde{A}_\xi^+ = \sqrt{2}M_\xi$.
Notice that, as the Dunkl operators are homogeneous of degree -1 *on polynomials, and since* M_ξ *are the multiplication operators, now obviously* $\widetilde{A}_\xi^-$ *and* $\widetilde{A}_\xi^+$ *are the lowering and the raising operators on* $\mathscr{P}(\mathbb{C}^N)$.

***Remark* 2.9.** The relationship between $\widetilde{T}_\xi(k)$ and $\widetilde{M}_\eta$, given in the statement (v) above, is the key commutation relation defining the so-called rational Cherednik algebra (cf. [E-G, p. 250]). Indeed, here we have an explicit action on a Hilbert space, with the relevant adjoints, representing this rational Cherednik algebra. After [B-Ø2] was finished, we were able to construct two operators, now acting in the tensor algebra, and which depend on a parameter $-1 \le q \le 1$. These two operators satisfy a q-commutation relation analogue to the one in (v). The details of such a q-deformation of $\mathscr{F}_k(\mathbb{C}^N)$ will appear in a forthcoming paper.

The above theorem, which is of independent interest, is mainly useful to obtain the quantum Calogero-Moser (CM) rational system in the Fock model. We refer to [B-Ø2] for more details on this matter. Let

$$\mathscr{L}_k := \Delta - 2\sum_{\alpha\in\mathscr{R}^+} \frac{1}{\langle\alpha, x\rangle^2} k_\alpha(k_\alpha - r_\alpha),$$

and consider the following gauge equivalent version

$$\mathscr{H}_k := \tfrac{1}{4}\omega_k^{-1/2}(-\mathscr{L}_k + 4\langle x, x\rangle)\omega_k^{1/2} = \tfrac{1}{4}(-\Delta_k + 4\langle x, x\rangle)$$

of the CM Hamiltonian with harmonic confinement and reflection terms. These operators are introduced by Heckman [H] to prove the quantum integrability of the G-invariant part of $\mathscr{H}_k$. See also [Ol-P].

Theorem 2.10. (cf. [B-Ø2]) *Let* $\{\xi_1, \ldots, \xi_N\}$ *be any orthonormal basis of* $\mathbb{C}^N$. *On* $\mathscr{F}_k(\mathbb{C}^N)$, *the corresponding operator to the Hamiltonian* $\mathscr{H}_k$ *is given by*

$$\widetilde{\mathscr{H}_k} = (\gamma_k + N/2) + \sum_{i=1}^N \xi_i \partial_{\xi_i},$$

where $\gamma_k = \sum_{\alpha\in\mathscr{R}^+} k_\alpha$.

The advantage of the above theorem is that the study of the Hamiltonian $\mathscr{H}_k$ in the Fock model is rather easy. In particular, by means of the $\mathscr{B}_k^{-1}$'s expression given in Remark 2.6(ii), we can recover in a simple way the Hermite polynomials and functions investigated independently in [R1].

We close this section by describing the structure of a representation of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ on $\mathscr{P}(\mathbb{C}^N)$. This representation, together with the left regular action of the Coxeter group G, allows to obtain the branching decomposition of the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ under the action of $G \times \widetilde{SL(2,\mathbb{R})}$. Those readers who are familiar with the theory of Howe reductive dual pairs [Ho2] will find that our formulation can be thought of as an analogue of this theory.

Choose $z_1, z_2, \ldots, z_N$ as the usual system of coordinates on $\mathbb{C}^N$. Let

$$\mathbb{E} = \frac{1}{2}\sum_{i=1}^N z_i^2, \quad \mathbb{F} = -\frac{1}{2}\Delta_k, \quad \mathbb{H} = N/2 + \gamma_k + \sum_{i=1}^N z_i\partial_{z_i}.$$

In the notation of Theorem 2.9, the operator $\mathbb{H} = \widetilde{\mathscr{H}_k}$. Then $\mathbb{E}$ (resp. $\mathbb{F}$) acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a creation (resp. annihilation) operator, and $\mathbb{H}$ acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a number operator. If $\mathscr{P}(\mathbb{C}^N) = \bigoplus_{m=0}^\infty \mathscr{P}_m(\mathbb{C}^N)$ is the natural grading on $\mathscr{P}(\mathbb{C}^N)$, it is clear that $\mathbb{E}$ raises

$\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m+2}(\mathbb{C}^N)$, $\mathbb{F}$ lowers $\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m-2}(\mathbb{C}^N)$, and $\mathbb{H}$ multiplies (elementwise) $\mathscr{P}_m(\mathbb{C}^N)$ by the number $(N/2+\gamma_k+m)$. In [H], Heckman showed the following commutation relations

$$[\mathbb{E},\mathbb{F}]=\mathbb{H},\quad [\mathbb{E},\mathbb{H}]=-2\mathbb{E},\quad [\mathbb{F},\mathbb{H}]=2\mathbb{F}. \tag{2.3}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (2.3) gives rise to an infinitesimally unitary representation ϖ_k of $\mathfrak{sl}(2,\mathbb{R})$. The unitarity of ϖ_k follows from the fact that $\mathbb{E}^*=-\mathbb{F}$ and $\mathbb{H}=\mathbb{H}^*$ (cf. [B-Ø2, Theorem 3.7]). Notice also that $\mathbb{H}$ has discrete spectrum bounded below.

On $\mathscr{P}(\mathbb{C}^N)$, the representation ϖ_k can be described as

$$\varpi_k(\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}})=\mathfrak{sl}_2^{(2,0)}\oplus\mathfrak{sl}_2^{(1,1)}\oplus\mathfrak{sl}_2^{(0,2)}, \tag{2.4}$$

where

$$\mathfrak{sl}_2^{(2,0)}=\mathrm{Span}\{\mathbb{E}\},\quad \mathfrak{sl}_2^{(1,1)}=\mathrm{Span}\{\mathbb{H}\},\quad \mathfrak{sl}_2^{(0,2)}=\mathrm{Span}\{\mathbb{F}\}.$$

The decomposition (2.4) is an instance of the Cartan decomposition

$$\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}=\mathfrak{p}^+\oplus\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{p}^-$$

where $\mathfrak{sl}_2^{(2,0)}\simeq\varpi_k(\mathfrak{p}^+)$, $\mathfrak{sl}_2^{(1,1)}\simeq\varpi_k(\mathfrak{k}_{\mathbb{C}})$, and $\mathfrak{sl}_2^{(0,2)}\simeq\varpi_k(\mathfrak{p}^-)$. Here $\mathfrak{k}=\mathfrak{u}(1)$, the Lie algebra of the compact group $U(1)$. The integrated form of the Lie algebra representation ϖ_k is an analogue of the metaplectic representation, or the oscillator representation, of the universal covering $\widetilde{SL(2,\mathbb{R})}$ of the group $SL(2,\mathbb{R})$. Notice that if $N/2+\gamma_k\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$, we obtain a unitary representation of the double covering $Mp(2,\mathbb{R})$ of $SL(2,\mathbb{R})$, and if $N/2+\gamma_k\in\mathbb{Z}$, we obtain a representation of $SL(2,\mathbb{R})$. By applying the Segal-Bargmann transform, one obtains the Schrödinger picture of this representation of $\widetilde{SL(2,\mathbb{R})}$. However, for our purpose, its infinitesimal action (2.4) is enough.

Since ϖ_k is an infinitesimally unitary representation, and the operator $\mathbb{H}$, which is the generator of $\mathfrak{k}$, has a positive spectrum, then the representation contains vectors v_0 such that $\varpi_k(\mathfrak{p}^-)v_0=0$ and $\varpi_k(\mathfrak{k})v_0=(m+N/2+\gamma_k)v_0$ for some positive integer m. The vector v_0 is the so-called lowest weight vector for a representation, and the number $(m+N/2+\gamma_k)$ is the lowest weight. Then the space of representation has an orthonormal basis consisting of vectors $v_\ell\in\varpi_k(\mathfrak{p}^+)^\ell v_0$. It is easy to check that each vector v_ℓ is an eigenvector for $\varpi_k(\mathfrak{k})$ with eigenvalue $(m+2\ell+N/2+\gamma_k)$. Denote by $\mathscr{W}_{m+N/2+\gamma_k}$ the unitary representation of $\widetilde{SL(2,\mathbb{R})}$ with lowest weight $m+N/2+\gamma_k$.

For $m\in\mathbb{N}$, set $\mathscr{H}_m(k)$ to be the space of harmonic homogeneous polynomials of degree m, i.e. all functions $p\in\mathscr{P}_m(\mathbb{C}^N)$ such that $\Delta_k p=0$. It is clear that $p\in\mathscr{H}_m(k)$ if and only if $\varpi_k(\mathfrak{k})p=(m+N/2+\gamma_k)p$ and $\varpi_k(\mathfrak{p}^-)p=0$.

Now one of the key features in this formalism is the following branching decomposition. We refer to [B-Ø2] for its proof, which was inspired by Sobolev's argument in the classical case [Sob]. The notation $[m/2]$ stands for the integer part of $m/2$.

Theorem 2.11. (cf. [B-Ø2]) *The space $\mathscr{P}_m(\mathbb{C}^N)$ of homogeneous polynomials of degree m has a unique decomposition of the form*

$$\mathscr{P}_m(\mathbb{C}^N)=\sum_{\mu=0}^{[m/2]}{}^{\oplus}\langle z,z\rangle^\mu\mathscr{H}_{m-2\mu}(k),$$

where $\mathscr{H}_{m-2\mu}(k)$ denotes the space of harmonic homogeneous polynomials of degree $m-2\mu$. Moreover, each homogeneous polynomial $p\in\mathscr{P}_m(\mathbb{C}^N)$ can be written in a unique way as

$$p(z)=\sum_{\mu=0}^{[m/2]}\frac{\Gamma(N/2+m-\mu+\gamma_k-1)}{4^\mu\Gamma(\mu+1)\Gamma(N/2+m+\gamma_k-1)}\langle z,z\rangle^\mu h_{m-2\mu}(z),$$

where $h_{m-2\mu} \in \mathscr{H}_{m-2\mu}(k)$ *and is given explicitly by*

$$h_{m-2\mu}(z) = \sum_{\nu=0}^{[m/2]-\mu} \frac{(-1)^\nu \Gamma(N/2+m-2\mu-\nu-1+\gamma_k)}{4^\nu \Gamma(\nu+1)\Gamma(N/2+m-2\mu+\gamma_k-1)} \langle z,z\rangle^\nu \Delta_k^{\mu+\nu} p(z).$$

For $g \in G$, denote by $\pi(g)$ the left regular action of G on $\mathscr{F}_k(\mathbb{C}^N)$

$$\pi(g)f(z) = f(g^{-1}z).$$

The actions of G and $\mathfrak{sl}(2,\mathbb{R})$ on $\mathscr{F}_k(\mathbb{C}^N)$ commute.

We now summarize the consequences of all the above computations and discussions in the light of Theorem 2.10.

Theorem 2.12. (cf. [B-Ø2]) *As a* $G \times \widetilde{SL(2,\mathbb{R})}$*-module, the Fock space admits the following multiplicity-free decomposition*

$$\mathscr{F}_k(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathscr{H}_m(k) \otimes \mathscr{W}_{m+N/2+\gamma_k}, \tag{2.5}$$

where $\mathscr{W}_{m+N/2+\gamma_k}$ *is the* $\widetilde{SL(2,\mathbb{R})}$*-representation with lowest weight* $m+N/2+\gamma_k$. *We also have the following separation of variables decomposition*

$$\mathscr{P}(\mathbb{C}^N) = \sum_{m=0}^{\infty}{}^{\oplus} \sum_{\mu=0}^{[m/2]}{}^{\oplus} \langle z,z\rangle^\mu \mathscr{H}_{m-2\mu}(k).$$

***Remark* 2.13.** (i) Recall that the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ was defined for non-negative multiplicity functions k. Now, notice that the right hand side of (2.5) exists for all $\gamma_k > -N/2$, where $\gamma_k = \sum_{\alpha\in\mathscr{R}^+} k_\alpha$, which implies that $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ could have negative-values up to a certain point. By analytic continuation, it follows that the left hand side of (2.5), i.e. the Fock space $\mathscr{F}_k(\mathbb{C}^N)$, exists also for these negative-valued multiplicity functions k.

(ii) The space $\mathscr{H}_m(k)$ is a unitary representation of G, in general not irreducible. It would be interesting to decompose it further.

3. Some applications of the $\mathfrak{sl}(2)$-triple

In this section we will give several applications of the $\mathfrak{sl}(2,\mathbb{R})$-representation discussed in the previous section. In the first and the second applications we adapt the method of R. Howe in the theory of ordinary derivatives, i.e. when $k \equiv 0$ (cf. [Ho1], [Ho-T]). In the last application we employ M. Vergne's approach for proving the Rossmann-Kirillov character formula [Ve].

I. *A Bochner formula for the Dunkl transform.* Denote by $\mathscr{S}(\mathbb{R}^N)$ the Schwartz space of rapidly decreasing functions equipped with the usual Fréchet space topology. From Theorem 2.11 it follows, by standard arguments, that

$$\mathscr{S}(\mathbb{R}^N) = \sum_{m=0}^{\infty}{}^{\oplus} \mathscr{H}_{m,\mathbb{R}}(k) \cdot \mathscr{I}(\mathbb{R}^N), \tag{3.1}$$

where $\mathscr{I}(\mathbb{R}^N)$ denotes the space of $O(N)$-invariant Schwartz functions on $\mathbb{R}^N$, and $\mathscr{H}_{m,\mathbb{R}}(k)$ is the space of harmonic homogeneous polynomials in $\mathscr{P}(\mathbb{R}^N)$ of degree m. By abuse of notation we will write $\mathscr{H}_m(k)$ for $\mathscr{H}_{m,\mathbb{R}}(k)$. In the light of (3.1) we may consider the map

$$\zeta_{m,k}^N : \mathscr{H}_m(k) \otimes \mathscr{S}(\mathbb{R}^+) \to \mathscr{S}(\mathbb{R}^N),$$

defined by

$$\zeta_{m,k}^N(h_m \otimes \psi)(x) := h_m(x)\psi(\|x\|^2), \qquad h_m \in \mathscr{H}_m(k), \quad \psi \in \mathscr{S}(\mathbb{R}^+).$$

By means of the representation ϖ_k we construct a representation $\pi^N_{m,k}$ of $\mathfrak{sl}(2,\mathbb{R})$ on $\mathscr{S}(\mathbb{R}^+)$ as follows

$$\zeta^N_{m,k}\big(h_m \otimes \pi^N_{m,k}(X)\psi\big) = \varpi_k(X)\big(\zeta^N_{m,k}(h_m \otimes \psi)\big), \qquad X \in \mathfrak{sl}(2,\mathbb{R})$$

for fixed $h_m \in \mathscr{H}_m(k)$. Using (2.4), one may check that

$$\begin{aligned}
\pi^N_{m,k}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= \frac{1}{2}t, \\
\pi^N_{m,k}\begin{bmatrix} 1 & 0 \\ 0 & \text{-1} \end{bmatrix} &= 2t\frac{d}{dt} + \big(m + \frac{N}{2} + \gamma_k\big), \\
\pi^N_{m,k}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= -2\Big\{t\frac{d^2}{dt^2} + \big(m + \frac{N}{2} + \gamma_k\big)\frac{d}{dt}\Big\},
\end{aligned}$$

where $\gamma_k = \sum_{\alpha\in\mathscr{R}^+} k_\alpha$, and t is the positive variable of $\mathbb{R}^+$. Note that $\pi^N_{m,k}$ does not depend on h_m, and depends only on $m + \frac{N}{2} + \gamma_k$.

In [B-Ø2, Corollary 4.6] we proved that the Dunkl transform $\mathscr{D}_k$ can be written as

$$\mathscr{D}_k = e^{i\frac{\pi}{2}(\gamma_k + N/2)} e^{-i\frac{\pi}{8}(-\Delta_k + 4\langle x,x\rangle)},$$

whilst $\mathbb{X} := \big\{\frac{1}{4}(-\Delta_k + 4\langle x,x\rangle)\big\}$ is the generator of the Lie algebra $\mathfrak{k} \cong \mathfrak{so}(2)$. Thus $\mathscr{D}_k$ is in $\widetilde{SL(2,\mathbb{R})}$, up to a constant. Hence we may define the transform $\mathscr{D}^N_{m,k}$ on $\mathscr{S}(\mathbb{R}^+)$ by

$$\begin{aligned}
\zeta^N_{m,k}\big(h_m \otimes \mathscr{D}^N_{m,k}(\psi)\big) &:= \zeta^N_{m,k}\big(h_m \otimes \pi^N_{m,k}(e^{i\frac{\pi}{2}(\gamma_k+N/2)}e^{-i\frac{\pi}{2}\mathbb{X}})\psi\big) \\
&= \mathscr{D}_k\big(\zeta^N_{m,k}(h_m \otimes \psi)\big).
\end{aligned}$$

This implies the following Bochner formula for the Dunkl transform.

Theorem 3.1. (i) *For $f(x) = h_m(x)\psi(\|x\|^2)$, with $h_m \in \mathscr{H}_m(k)$ and $\psi \in \mathscr{S}(\mathbb{R}^+)$, we have*

$$\mathscr{D}_k(f)(\xi) = h_m(\xi)\mathscr{D}^N_{m,k}(\psi)(\|\xi\|^2),$$

where $\mathscr{D}^N_{m,k}$ depends only on $m + \frac{N}{2} + \gamma_k$, up to a constant, i.e.

$$e^{-i\frac{\pi}{2}(\gamma_k + \frac{N}{2})}\mathscr{D}^N_{m,k} = e^{-i\frac{\pi}{2}(\gamma_{k'} + \frac{N'}{2})}\mathscr{D}^{N'}_{m',k'}$$

if

$$m + \frac{N}{2} + \gamma_k = m' + \frac{N'}{2} + \gamma_{k'}. \tag{3.2}$$

(ii) *The transform $\mathscr{D}^N_{m,k}$ coincides with the usual Hankel transform. More precisely, for $\psi \in \mathscr{S}(\mathbb{R}^+)$*

$$\mathscr{D}^N_{m,k}(\psi)(r^2) = e^{-i\frac{\pi}{2}m}\mathscr{H}_{m+\frac{N}{2}+\gamma_k-1}(\psi\circ\Upsilon)(r),$$

where $\Upsilon(t) := t^2$ for $t \in \mathbb{R}$, and

$$\mathscr{H}_\nu f(r) := \int_0^\infty f(s)\frac{J_\nu(rs)}{(rs)^\nu}s^{2\nu+1}ds$$

denotes the Hankel transform, with J_ν is the Bessel function of the first kind. In these circumstances, (i) reads

$$\mathscr{D}_k\big(h_m\psi(\|\cdot\|)\big)(\xi) = e^{-i\frac{\pi}{2}m}h_m(\xi)\mathscr{H}_{m+\gamma_k+\frac{N}{2}-1}(\psi)(\|\xi\|).$$

To prove the statement (ii) above, we start with the case $m = 0$, and then we use (3.2) to deduce the claim for general m. The above theorem generalizes [D2, Theorem 2.1], as the example below shows.

***Example* 3.2.** (Hecke-type formula) If we choose $\psi(s)=e^{-\frac{s^2}{2}}$, then the following Hecke-type formula for the Dunkl transform holds

$$\begin{aligned}\mathscr{D}_k\big(e^{-\frac{\|x\|^2}{2}}h_m\big)(\xi) &= e^{-i\frac{\pi}{2}m}h_m(\xi)\mathscr{H}_{m+\gamma_k+\frac{N}{2}-1}\big(e^{-\frac{s^2}{2}}\big)(\|\xi\|)\\ &= e^{-i\frac{\pi}{2}m}e^{-\frac{\|\xi\|^2}{2}}h_m(\xi).\end{aligned}$$

II. *Huygens' principle.* It is well known that propagation of waves is different in the two- and in the three-dimensional spaces. For instance, suppose we make a "noise" located near a point x at time $t=0$. Thus we can "hear" this noise at a point y at a later time t only if the distance $y-x$ from y to x is less then t. This phenomena holds in all dimensions, but something special happens in the three dimensional space. After the noise is heard, it moves away and leaves no vibration. This is the so-called Huygens principle. In mathematical terms, Huygens' principle can be explained as following. On $\mathbb{R}^{N+1}$, consider the classical wave equation $(\star)$ $\Delta u(x,t)=\partial_{tt}u(x,t)$. For odd $N\geq 3$, the solution of the Cauchy problem for $(\star)$ at every given point x_0 depends only on its Cauchy data in an arbitrary neighborhood on the light cone surface with vertex x_0. The problem of classifying all second order differential operators which satisfy Huygens' principle is still far from being fully solved (see for instance [L-St], [Co-Hi], [La-P], [Ø], [M], [He2], [Be-V], [Br-Ó-S], [C-F-V], [B]). In the present application, we will investigate the validity of Huygens' principle for $(\star)$ when Δ is replaced by the Dunkl-Laplacian operator Δ_k. We refer to [B-Ø3] for a complete investigation.

For a multiplicity function $k\in\mathscr{K}^+$, consider the following Cauchy problem for the wave equation associated with the Dunkl Laplacian operator

$$\begin{aligned}\Delta_k u_k(x,t)=\partial_{tt}u_k(x,t),\quad (x,t)\in\mathbb{R}^N\times\mathbb{R},\\ u_k(x,0)=f(x),\ \ \partial_t u_k(x,0)=g(x).\end{aligned}\tag{3.3}$$

To simplify the presentation of our results, we shall assume that f and g are smooth and supported in a closed ball of radius $R>0$ about the origin. We mention that in [B-Ø3], we investigated the Cauchy problem (3.3) where the Cauchy data (f,g) belong to the Schwartz space $\mathscr{S}(\mathbb{R}^N)$.

A standard argument shows that the solution of the Cauchy problem (3.3) is uniquely given by

$$u_k(x,t)=(P^{(1)}_{k,t}*_k f)(x)+(P^{(2)}_{k,t}*_k g)(x),$$

where, for fixed t,

$$P^{(1)}_{k,t}=\mathscr{D}_k^{-1}\big[\cos(t\|\cdot\|)\big],\quad P^{(2)}_{k,t}=\mathscr{D}_k^{-1}\big[\sin(t\|\cdot\|)/\|\cdot\|\big].$$

Here $*_k$ is a generalized translation, and when $k\equiv 0$, $*_0$ coincides with the usual Euclidean convolution. We refer to [T] for the definition of $*_k$. In terms of the propagators, Huygens' principle amounts to the fact that $P^{(1)}_{k,t}$ and $P^{(2)}_{k,t}$ are supported on the light cone $\mathscr{C}:=\big\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid \|x\|^2-t^2=0\big\}$.

Choose $x_1,x_2,\ldots,x_N$ as the usual system of coordinates on $\mathbb{R}^N$. Let

$$\mathbb{E}_{N,1}:=\frac{1}{2}(\|x\|^2-t^2),\quad \mathbb{F}_{N,1}:=-\frac{1}{2}(\Delta_k-\partial_{tt}),$$
$$\mathbb{H}_{N,1}:=\frac{N+1}{2}+\gamma_k+\sum_{j=1}^N x_j\partial_j+t\partial_t.$$

Using (2.3), one may check the following commutation relations

$$[\mathbb{E}_{N,1},\mathbb{H}_{N,1}]=-2\mathbb{E}_{N,1},\quad [\mathbb{F}_{N,1},\mathbb{H}_{N,1}]=2\mathbb{F}_{N,1},\quad [\mathbb{E}_{N,1},\mathbb{F}_{N,1}]=\mathbb{H}_{N,1}.\tag{3.4}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (3.4) gives rise to a representation Ω_k of $\mathfrak{sl}(2,\mathbb{R})$, which could be described in a similar way to ϖ_k from the previous section.

Define the distributions $P_k^{(\ell)}$ on $\mathbb{R}^{N+1}$ by

$$P_k^{(\ell)}(\psi_1 \otimes \psi_2) := \int_{\mathbb{R}} P_{k,t}^{(\ell)}(\psi_1)\psi_2(t)dt, \qquad \ell = 1, 2,$$

where $\psi_1 \in \mathscr{S}(\mathbb{R}^N)$ and $\psi_2 \in \mathscr{S}(\mathbb{R})$. Thus we may rewrite the solution u_k as

$$u_k = P_k^{(1)} *_{k,x} f + P_k^{(2)} *_{k,x} g,$$

where $*_{k,x}$ is the $*_k$-convolution with respect to x. Since $\mathscr{C}$ is the locus of zeros of $\|x\|^2 - t^2$, then $P_k^{(\ell)}$ ($\ell = 1, 2$) is supported on the light cone $\mathscr{C}$ if and only if

$$(\|x\|^2 - t^2)^m P_k^{(\ell)} = 0, \text{ i.e. } \mathbb{E}_{N,1}^m \cdot P_k^{(\ell)} = 0,$$

for some positive integer m. Further, one can prove that

$$(\Delta_k - \partial_{tt})P_k^{(\ell)} = 0, \text{ i.e. } \mathbb{F}_{N,1} \cdot P_k^{(\ell)} = 0. \tag{3.5}$$

These two facts yield the following theorem.

Theorem 3.3. (cf. [B-Ø3]) *Huygens' principle holds in the sense that $P_k^{(\ell)}$ ($\ell = 1, 2$) is supported on the light cone $\mathscr{C}$ if and only if $P_k^{(\ell)}$ generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$-module.*

In the light of the above theorem, we will look for conditions on N and k which may assure the existence of a finite dimensional representation. As a first step in this direction, when $(N+1)/2+\gamma_k \notin \mathbb{Z}$, Huygens' principle fails. This is due to the fact that the spectrum of the element $\mathbb{H}_{N+1} = (N+1)/2 + \gamma_k + \sum_{i=1}^{N+1} x_i\partial_i$ (or its dual) acting on $\mathscr{S}(\mathbb{R}^{N+1})$ (or $\mathscr{S}'(\mathbb{R}^{N+1})$) is $(N+1)/2 + \gamma_k + \mathbb{Z}$, whilst the spectrum of $\mathbb{H}_{N+1}$ (or its dual) in finite dimensional modules is contained in $\mathbb{Z}$. We refer to [B-Ø3] for more details on this matter.

This leaves the possibility that Huygens' principle may hold when $(N+1)/2 + \gamma_k \in \mathbb{Z}$, which turns out to be true. To see this, let us first define the dilation operator S_λ on $\mathscr{S}(\mathbb{R}^{N+1})$ by $S_\lambda\psi(x,t) = \psi(\lambda x, \lambda t)$, for $\lambda > 0$. By duality, S_λ acts on distributions in the standard way. Thus, we prove that

$$S_\lambda P_k^{(\ell)} = \lambda^\ell P_k^{(\ell)}.$$

Second, we introduce what we call the Dunkl-Fourier transform

$$\mathscr{D}_k\mathscr{F}\psi(x,t) := (2\pi)^{-1/2}c_k^{-1}\int_{\mathbb{R}^{N+1}} \psi(y,s)E_k(-ix,y)e^{ist}\omega_k(y)dyds,$$

for $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$. Here we used notation from the previous section. The transform $\mathscr{D}_k\mathscr{F}$ acts on distributions in the standard way. In particular, we show that

$$(\|x\|^2 - t^2)\mathscr{D}_k\mathscr{F}(P_k^{(\ell)}) = 0, \text{ i.e. } \mathbb{E}_{N,1} \cdot \mathscr{D}_k\mathscr{F}(P_k^{(\ell)}) = 0,$$

and

$$S_\lambda\big(\mathscr{D}_k\mathscr{F}(P_k^{(\ell)})\big) = \lambda^{2\gamma_k+N+1-\ell}\mathscr{D}_k\mathscr{F}(P_k^{(\ell)}).$$

As a consequence of the above discussions, and in the light of (3.5), the following theorem holds.

Theorem 3.4. (cf. [B-Ø3]) *Under the assumption*

$$\frac{N+1}{2} + \gamma_k - \ell \in \mathbb{N}, \tag{3.6}$$

the tempered distribution $P_k^{(\ell)}$ generates an $\mathfrak{sl}(2,\mathbb{R})$-module of dimension

$$d(k,\ell)=\frac{N+3}{2}+\gamma_k-\ell,$$

with highest weight vector $\mathscr{D}_k\mathscr{F}(P_k^{(\ell)})$ of highest weight $\left(\frac{N+1}{2}+\gamma_k-\ell\right)$. Further, for each ℓ there exists a constant α_ℓ such that

$$P_k^{(\ell)}=\alpha_\ell\,\mathbb{F}_{N,1}^{d(k,\ell)-1}\cdot\mathscr{D}_k\mathscr{F}(P_k^{(\ell)}).$$

By taking into account the condition (3.6) for both $P_k^{(1)}$ and $P_k^{(2)}$, and recalling Theorem 3.3, we can state Huygens' principle for the Cauchy problem (3.3) as following.

Theorem 3.5. (cf. [B-Ø3]) *Let u_k be the solution of the Cauchy problem (3.3) and suppose that the Cauchy data (f,g) are supported inside the closed ball of radius $R>0$ about the origin. The support of u_k is contained in the conical shell*

$$\mathscr{C}=\left\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid |t|-R\leq\|x\|\leq|t|+R\right\}\tag{3.7}$$

if and only if

$$(N-3)/2+\gamma_k\in\mathbb{N}.$$

The shell $\mathscr{C}$ is the union

$$\bigcup_{\|y\|\leq R}\mathscr{C}_y\tag{3.8}$$

where $\mathscr{C}_y$ is the light cone

$$\mathscr{C}_y=\left\{(x,t)\mid \|x-y\|=|t|\right\}.$$

III. ***A Harish-Chandra restriction type theorem for the Dunkl transform.*** Let G be a connected noncompact semi-simple Lie group with finite center, and let K be a maximal compact subgroup of G. The symmetric space G/K is a Riemannian symmetric space of the noncompact type. Let θ be the Cartan involution on G corresponding to K. We use the same symbol θ for its differential on the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Denote by $\Sigma=\Sigma(\mathfrak{g},\mathfrak{a})$ the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. For every $\alpha\in\Sigma$ let $\mathfrak{g}^{(\alpha)}$ be the corresponding root space, and set $m_\alpha=\dim(\mathfrak{g}^{(\alpha)})$. For $\alpha\in\Sigma$, define $H_\alpha\in[\mathfrak{g}^{(\alpha)},\mathfrak{g}^{(-\alpha)}]$ by $\alpha(H_\alpha)=2$, and recall the definition of the reflections r_α for $\alpha\in\Sigma$. Consider the Coxeter group $W:=\langle r_\alpha\mid\alpha\in\Sigma\rangle$. Usually W is called the Weyl group of Σ. Henceforth, we shall choose a positive root system Σ^+ in Σ.

For $X,Y\in\mathfrak{g}$, set

$$\langle\!\langle X,Y\rangle\!\rangle:=-B(X,\theta(Y)),$$

where B is the Killing form of $\mathfrak{g}$. Then $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is an inner product on $\mathfrak{g}$. As $\mathfrak{a}\subset\mathfrak{p}$, the Killing form $B(\cdot,\cdot)$ and the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ agree on $\mathfrak{a}$. We use this inner product to identify $\mathfrak{a}$ and $\mathfrak{a}^*$, and we shall denote the norm on $\mathfrak{a}$ by $\|\cdot\|$. Indeed, if $N=\dim(\mathfrak{a})$, then $(\mathfrak{a},\langle\!\langle\cdot,\cdot\rangle\!\rangle)\cong(\mathbb{R}^N,\langle\cdot,\cdot\rangle)$.

For a function f on $\mathfrak{p}$, we denote by $\mathscr{M}f$ the function of $\mathfrak{a}$ defined by

$$\mathscr{M}f(X)=\int_K f(\mathrm{Ad}(k)X)dk.$$

We normalize the Haar measure dk on K such that

$$\int_{\mathfrak{p}}f(Y)dY=\int_K\int_{\mathfrak{a}}f(\mathrm{Ad}(k)X)\prod_{\alpha\in\Sigma^+}|\alpha(X)|^{m_\alpha}dkdX,\tag{3.9}$$

where the Lebesgue measures on $\mathfrak{p}$ and $\mathfrak{a}$ will be normalized as below. (See [He1] for the integral formula above.) Since K is compact, the integral $\mathscr{M}$ is continuous, from the Schwartz space $\mathscr{S}(\mathfrak{p})$ to the space $\mathscr{S}(\mathfrak{a})^W$ of W-invariant Schwartz functions on $\mathfrak{a}$.

For a function f on $\mathfrak{p}$, we define its $\mathfrak{p}$-Fourier transform $\mathscr{F}_{\mathfrak{p}}(f)$ by

$$\mathscr{F}_{\mathfrak{p}} f(X) = \int_{\mathfrak{p}} f(Y) e^{-i\langle\langle X,Y\rangle\rangle} dY.$$

We normalize the Lebesgue measure on $\mathfrak{p}$ by $\mathscr{F}_{\mathfrak{p}}^2(f)(X) = f(-X)$. We extend $\mathscr{F}_{\mathfrak{p}}$ to a transform of tempered distributions in the usual way. The crucial observation is that $\mathscr{F}_{\mathfrak{p}}$ belongs, up to a scalar multiple, to the one parameter group of unitary transforms $e^{-it(B_{\mathfrak{p}}-\Delta_{\mathfrak{p}})}$. Here $B_{\mathfrak{p}}$ is the restriction of the Killing form B to $\mathfrak{p}\times\mathfrak{p}$, and $\Delta_{\mathfrak{p}}$ is the Laplace operator on $\mathfrak{p}$. More precisely

$$\mathscr{F}_{\mathfrak{p}} = e^{i\frac{\pi}{4}\dim(\mathfrak{p})} e^{-i\frac{\pi}{4}\left(B_{\mathfrak{p}}-\Delta_{\mathfrak{p}}\right)}. \tag{3.10}$$

See for instance [Ve] for a general discussion.

Now we turn our attention to the Dunkl transform in the present setting. For $H \in \mathfrak{a}$, let $\partial(H)$ denote the corresponding directional derivative in $\mathfrak{a}$. The Dunkl operator takes the form

$$T_H(m)f(X) = \partial(H)f(X) + \sum_{\alpha\in\Sigma^+} \frac{m_\alpha}{2}\alpha(H)\Big(\frac{f(X)-f(r_\alpha X)}{\alpha(X)}\Big),$$

and the Dunkl-Laplacian operator is given by

$$\Delta_m f(X) = \sum_{j=1}^{N} \partial(H_j)^2 f(X) + \sum_{\alpha\in\Sigma^+} m_\alpha\Big(\frac{\partial_\alpha f(X)}{\alpha(X)} - \frac{f(X)-f(r_\alpha X)}{\alpha(X)^2}\Big),$$

where $\{H_j\}_{j=1}^N$ is an orthonormal basis of $\mathfrak{a}$, and $\partial_\alpha = \partial(H_\alpha)$. Further, we may rewrite the Dunkl transform $\mathscr{D}_m$ (instead of the notation $\mathscr{D}_k$) as

$$\mathscr{D}_m f(H) = c_m^{-1} 2^{-\gamma_m - N/2} \int_{\mathfrak{a}} f(X) E_m(-iX, H) \prod_{\alpha\in\Sigma^+} |\alpha(X)|^{m_\alpha} dX,$$

where $\gamma_m = \sum_{\alpha\in\Sigma^+} m_\alpha/2$, and $c_m = \int_{\mathfrak{a}} e^{-\|X\|^2} \prod_{\alpha\in\Sigma^+} |\alpha(X)|^{m_\alpha} dX$. Here the Lebesgue measure on $\mathfrak{a}$ is normalized so that $\mathscr{D}_m^2 f(X) = f(-X)$. We mention that $\mathscr{D}_m$ leaves the Schwartz space $\mathscr{S}(\mathfrak{a})$ stable.

Now recall the $\mathfrak{sl}(2,\mathbb{R})$-triple $\{\mathbb{E},\mathbb{F},\mathbb{H}\}$ from Section 2. By [B-Ø2, Corollary 4.6] the Dunkl transform can be written as

$$\mathscr{D}_m = e^{i\frac{\pi}{2}(\gamma_m+N/2)} e^{-i\frac{\pi}{4}\left(\|X\|^2-\Delta_m\right)} = e^{i\frac{\pi}{2}(\gamma_m+N/2)} e^{-i\frac{\pi}{2}(\mathbb{E}+\mathbb{F})}. \tag{3.11}$$

To establish the connection between $\mathscr{F}_{\mathfrak{p}}$ and the Dunkl transform, one needs the following fundamental properties of the map $\mathscr{M}$, which are rather clear (recall that $\mathscr{M}$ is a W-invariant map).

Lemma 3.6. *On the Schwartz space $\mathscr{S}(\mathfrak{p})$, the map $\mathscr{M}$ satisfies*

$$\mathscr{M} B_{\mathfrak{p}}(\cdot,\cdot) = \|\cdot\|^2 \mathscr{M}, \qquad \mathscr{M}\Delta_{\mathfrak{p}} = \Delta(m)\mathscr{M}.$$

Below we state the main result which follows from (3.10) and (3.11) by means of the previous lemma. Note that $\dim(\mathfrak{p}) = 2\gamma_m + N$.

Theorem 3.7. *For $f \in \mathscr{S}(\mathfrak{p})$, we have*

$$\mathscr{M}\mathscr{F}_{\mathfrak{p}}(f) = \mathscr{D}_m \mathscr{M}(f).$$

The above theorem releases the connection between the Fourier analysis on the flat symmetric space $\mathfrak{p}$ and the Dunkl theory on $\mathfrak{a}$.

Denote by $\mathscr{S}'(\mathfrak{p})$ (resp. $\mathscr{S}'(\mathfrak{a})$) the space of tempered distributions on $\mathfrak{p}$ (resp. $\mathfrak{a}$). Let $\mathscr{S}'(\mathfrak{p})^K$ be the space of K-invariant elements in $\mathscr{S}'(\mathfrak{p})$. The transpose $\mathscr{F}_{\mathfrak{p}}^t$ of $\mathscr{F}_{\mathfrak{p}}$ leaves $\mathscr{S}'(\mathfrak{p})$ and $\mathscr{S}'(\mathfrak{p})^K$ stable. Similarly, $\mathscr{D}_m^t$ leaves $\mathscr{S}'(\mathfrak{a})$ and $\mathscr{S}'(\mathfrak{a})^W$ stable, and the transpose $\mathscr{M}^t$ of $\mathscr{M}$ maps $\mathscr{S}'(\mathfrak{a})^W$ to $\mathscr{S}'(\mathfrak{p})^K$. Moreover, by means of the integral formula (3.9), it follows that if $f \in \mathscr{S}'(\mathfrak{a})^W$ is a function, then

$$\mathscr{M}^t f(p) = \frac{f(X)}{\prod_{\alpha\in\Sigma^+} |\alpha(X)|^{m_\alpha}}, \qquad \forall\ p = \mathrm{Ad}(k)X \in \mathfrak{p},\ \text{with } k \in K \text{ and } X \in \mathfrak{a}.$$

For $X \in \mathfrak{a}$, denote by δ_X the distribution on $\mathfrak{p}$ defined by $\langle \delta_X, f\rangle = \int_K f(\mathrm{Ad}(k)X)dk$, and define δ_X° on $\mathfrak{a}$ by $\langle \delta_X^\circ, \mathscr{M}f\rangle = \langle \delta_X, f\rangle$. Thus one can check that

$$\mathscr{M}^t \mathscr{D}_m^t(\delta_X^\circ)(Y) = c_m^{-1} 2^{-\gamma_m - N/2} \frac{1}{|W|} \sum_{w\in W} E_m(-iwX, Y), \qquad X, Y \in \mathfrak{a}.$$

As a direct application of Theorem 3.7, we can express the spherical functions on the flat symmetric space $\mathfrak{p}$ in terms of the Dunkl kernels. This explains the meaning of the Dunkl kernels in the context of Euclidean symmetric spaces.

Corollary 3.8. *The restriction of the spherical function*

$$\psi(X, Y) = \int_K e^{-iB(\mathrm{Ad}(k)X, Y)} dk, \qquad X \in \mathfrak{a},\ Y \in \mathfrak{p},$$

to $\mathfrak{a} \times \mathfrak{a}$ *coincides with*

$$c_m^{-1} 2^{-\gamma_m - N/2} \frac{1}{|W|} \sum_{w\in W} E_m(-iwX, Y).$$

***Remark* 3.9.** (i) The Bessel functions $J_m(X, Y) := \frac{1}{|W|}\sum_{w\in W} E_m(wX, Y)$ were first introduced by Opdam independently from the context of flat symmetric spaces [O]. See also [B-Ø1], where we denote $J_m(X, Y)$ by $F^\circ(X, m, Y)$.

(ii) The above corollary was also proved by de Jeu in [J] using a different approach. De Jeu's method is based on a result due to Heckman, together with Helgason's result regarding the uniqueness of the solution $\psi(\cdot, Y)$ to the equation $\partial(p)\psi(X, Y) = p(Y)\psi(X, Y)$, for $p \in S(\mathfrak{p})^K$, such that $\psi(0, Y) = 1$.

As a consequence of Corollary 3.8, we may express the average of the Dunkl kernels over the Weyl group in terms of the exponential functions. Indeed, let U be a connected, simply connected, compact, semi-simple Lie group. Let $\mathfrak{u} = \mathrm{Lie}(U)$, and $\mathfrak{g}$ be the complexification of $\mathfrak{u}$. Let T be a maximal torus of U, and $\mathfrak{t}_0$ be its Lie algebra. Then the subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ generated by $\mathfrak{t}_0$ is a Cartan subalgebra. Denote by $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$. Using the Harish-Chandra integral formula, the following holds for all $X, Y \in \mathfrak{t}$

$$\frac{1}{|W(\mathfrak{g},\mathfrak{t})|} \sum_{w\in W(\mathfrak{g},\mathfrak{t})} E_m(X, wY) = c_m 2^{\gamma_m + N/2} \prod_{\alpha\in\Delta^+} \alpha(\rho) \frac{\sum_{w\in W(\mathfrak{g},\mathfrak{t})} \det(w) e^{\langle\!\langle X, wY\rangle\!\rangle}}{\prod_{\alpha\in\Delta^+}\alpha(X) \prod_{\alpha\in\Delta^+}\alpha(Y)}, \tag{3.12}$$

where $W(\mathfrak{g}, \mathfrak{t}) := \langle r_\alpha \mid \alpha \in \Delta(\mathfrak{g}, \mathfrak{t})\rangle$, and $\rho = \sum_{\alpha\in\Delta^+} \alpha/2$.

***Remark* 3.10.** The argument presented in this subsection could also be generalized for an arbitrary symmetric pair (G, H) with a Lie group G, for which there is an involution σ of G such that $G_e^\sigma \subset H \subset G^\sigma$. Here G^σ is the subgroup of fixed points for σ, and G_e^σ denotes its identity component. We briefly outline this generalization. Let θ be the Cartan involution on G with the corresponding maximal compact subgroup K, such that $\theta\sigma = \sigma\theta$. We shall use the same symbol for an involution on G and its differential on the Lie algebra $\mathfrak{g}$. As usual, write $\mathfrak{k} = \mathfrak{g}^\theta$, $\mathfrak{p} = \mathfrak{g}^{-\theta}$, $\mathfrak{h} = \mathfrak{g}^\sigma$, and $\mathfrak{q} = \mathfrak{g}^{-\sigma}$. Let $\mathfrak{a}_q$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. We write $\mathfrak{g}^+ := \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q}$ and $\mathfrak{g}^- := \mathfrak{k}\cap\mathfrak{q} \oplus \mathfrak{p}\cap\mathfrak{h}$. In particular, $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^+$. Consider the following assumptions:

$$\begin{array}{l}\text{(i) } \Sigma(\mathfrak{g}, \mathfrak{a}_q) \text{ is the disjoint union of } \Sigma(\mathfrak{g}^+, \mathfrak{a}_q) \text{ and } \Sigma(\mathfrak{g}^-, \mathfrak{a}_q),\\ \text{(ii) the multiplicity of each } \alpha \in \Sigma(\mathfrak{g}^-, \mathfrak{a}_q) \text{ equals } 2.\end{array} \tag{\textbf{H}}$$

Under the hypothesis (**H**) one can prove that

$$\mathscr{M}_q f(X) := \prod_{\alpha\in\Sigma^+(\mathfrak{g}^-,\mathfrak{a}_q)} |\alpha(X)| \int_H f(\mathrm{Ad}(h)X)dh, \qquad X \in \mathfrak{a}_q,$$

intertwines the Fourier transform on $\mathfrak{q}$ with the Dunkl transform on $\mathfrak{a}_q$. In particular, for $X, Y \in \mathfrak{a}_q$, one can express $\int_H e^{-iB(\mathrm{Ad}(h)X,Y)}dh$, as a distribution, in terms of the Dunkl kernels related to the Riemannian symmetric space $G^+/K\cap H$, divided by the product $\prod_{\alpha\in\Sigma^+(\mathfrak{g}^-,\mathfrak{a}_q)}\alpha(X)\alpha(Y)$. Unfortunately we were not able to find an interesting example where the hypothesis (**H**) holds, other than the cases of Riemannian symmetric spaces (treated in this subsection), and the group cases investigated earlier by Rossmann [Ro] and Vergne [Ve]. However, in the group cases, we can apply the approach outlined above to the c-dual pair $(\mathfrak{g}^c,\mathfrak{h})$ of $(\mathfrak{g},\mathfrak{h})$ where $\mathfrak{g}^c = \mathfrak{h}\oplus i\mathfrak{q} = \mathfrak{k}^c\oplus\mathfrak{p}^c$ with $\mathfrak{k}^c = \mathfrak{h}\cap\mathfrak{k}\oplus i(\mathfrak{q}\cap\mathfrak{p})$ and $\mathfrak{p}^c = \mathfrak{h}\cap\mathfrak{p}\oplus i(\mathfrak{q}\cap\mathfrak{k})$. Thus we can express the Fourier transform of the measures on orbits in the coadjoint representation in terms of the Dunkl kernels divided by products of noncompact roots. Now by means of (3.12) we recover Rossmann's explicit formula for the characters of discrete series representations.

References

[Ba] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform, Part I,* Comm. Pure Appl. Math. **14** (1961), 187–214

[Be-V] Y. Berest and A.P. Veselov, *Huygens' principle and integrability,* Russian Math. Surveys **49** (1994), 5–77

[Br-Ó-S] T. Branson, G. Ólafsson, and H. Schlichtkrull, *Huyghens' Principle in Riemannian Symmetric Spaces,* Math. Ann. **301** (1995), 445-462

[B] S. Ben Saïd, *Huygen's principle for the wave equation associated with the trigonometric Dunkl-Cherednik operators*, to appear in Math. Res. Lett. (2005)

[B-Ø1] S. Ben Saïd and B. Ørsted, *Bessel functions for root systems via the trigonometric setting,* Int. Math. Res. Not. **9** (2005), p.551–585

[B-Ø2] ———, *Segal-Bargmann transforms associated with Coxeter groups,* to appear in Math. Ann. (2005)

[B-Ø3] ———, *Wave equations for Dunkl differential-difference operators,* to appear in Indag. Math. (2005)

[C-F-V] O.A. Chalykh, M.V. Feigin, and A.P. Veselov, *New integrable generalizations of the Calogero-Moser quantum problem,* J. Math. Phys. **39** (1998), 695–703

[Ch] F.M. Cholewinski, *Generalized Fock spaces and associated operators,* SIAM J. Math. Anal. **15** (1984), 177–202

[Co-Hi] R. Courant and D. Hilbert, *Methods of mathematical physics. Vo. II: Partial differential equations,* Interscience Publishers, New York-London, 1962

[Da-Ó-Z] M. Davidson, G. Ólafsson, and G. Zhang, *Segal-Bargmann transform on Hermitian symmetric spaces,* J. Funct. Anal. **204** (2003), 157–195

[D1] C.F. Dunkl, *Differential-difference operators associated to reflection groups,* Trans. Amer. Math. Soc. **311** (1989), 167–183

[D2] ———, *Hankel transforms associated to finite reflection groups,* Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991), 123–138, Contemp. Math. **138**, Amer. Math. Soc., Providence, RI, 1992

[E-G] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism,* Invent. Math. **147** (2002), 243–348

[Fo] V. Fock, *Verallgemeinerung und Lösung der Diracschen statischen Gleichung,* Z. für Phys. **49** (1928), 339–357

[Fol] G.B. Folland, *Harmonic analysis in phase space,* Princeton University Press, Princeton, N.J., 1989

[Gr-M] L. Gross and P. Malliavin, *Hall's transform and the Segal-Bargmann map*, In: "Itô's stochastic calculus and probability theory", 73–116, Springer, Tokyo, 1996

[Ha] B.C. Hall, *The Segal-Bargmann "Coherent State" transform for compact Lie groups,* J. Funct. Anal. **112** (1994), 103–151

[He1] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces,* Academic Press, Inc., New York-London, 1978

[He2] ———, *Huygens' principle for wave equations on symmetric spaces*, J. Funct. Anal. **107** (1992), 279–288

[H] G.J. Heckman, *A remark on the Dunkl differential-difference operators,* Harmonic analysis on reductive groups (Brunswick, ME, 1989), 181–191, Progr. Math. **101**, Birkhäuser Boston, Boston, MA, 1991

[Ho1] R. Howe, *On the role of the Heisenberg group in harmonic analysis,* Bull. Amer. Math. Soc. **3** (1980), 821–843

[Ho2] ———, *Remarks on classical invariant theory,* Trans. Amer. Math. Soc. **313** (1989), 539–570

[Ho-T] R. Howe and E.C. Tan, *Nonabelian harmonic analysis. Applications of* $\mathrm{SL}(2,\mathbb{R})$, Universitext. Springer-Verlag, New York, 1992

[J] M.F. de Jeu, *The Dunkl transform,* Invent. Math. **113** (1993), 147–162

[L-St] J.E. Lagnese and K.L. Stellmacher, *A method of generating classes of Huygens's operators,* J. Math. Mech. **17** (1967), 461–472

[La-P] P.D. Lax and R.S. Phillips, *Scattering theory,* Academic Press, New York and London (1978)

[M] R.G. McLenaghan, *Huygens' principle*, Ann. Inst. H. Poincaré Sect. A (N.S.) **37** (1982), 211–236

[Ol-P] M. A. Olshanetsky and A. M. Perelomov, *Completely integrable Hamiltonian systems connected with semisimple Lie algebras,* Invent. Math. **37** (1976), 93–108

[O] E. Opdam, *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group,* Compositio Math. **85** (1993), 333–373

[Ó-Ø] G. Ólafsson and B. Ørsted, *Generalization of the Bargmann transform,* Lie theory and its applications in physics (Clausthal, 1995), World Sci. Publishing, River Edge, N.J. (1996), 3–14

[Ø] B. Ørsted, *The conformal invariance of Huygens' principle,* J. Differential Geom. **16** (1981), 1–9

[Ro] W. Rossmann, *Kirillov's character formula for reductive Lie groups,* Invent. Math. **48** (1978), 207–220

[R1] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators,* Comm. Math. Phys. **192** (1998), 519–542

[R2] ———, *Contributions to the theory of Dunkl operators,* Habilitations Thesis, TU München (1999)

[Seg] I.E. Segal, *Mathematical problems of relativistic physics,* Amer. Math. Soc., Providence, R.I., 1963

[Sob] S.L. Sobolev, *Introduction to the theory of cubature formulas,* Izdat. Nauka, Moscow, 1974

[So] F. Soltani, *Generalized Fock spaces and Weyl commutation relations for the Dunkl kernel,* Pacific J. Math. **214** (2004), 379–397

[Si-So] M. Sifi and F. Soltani, *Generalized Fock spaces and Weyl relations for the Dunkl kernel on the real line,* J. Math. Anal. Appl. **270** (2002), 92–106

[T] K. Trimèche, *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators,* Integral Transforms Spec. Funct. **13** (2002), 17–38

[Ve] M. Vergne, *On Rossmann's character formula for discrete series*, Invent. Math. **54** (1979), 11–14.

[Z] G. Zhang, *Branching coefficients of holomorphic representations and Segal-Bargmann transform,* J. Funct. Anal. **195** (2002), 306–349

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